Cryptographic Reductions By Bi-Deduction

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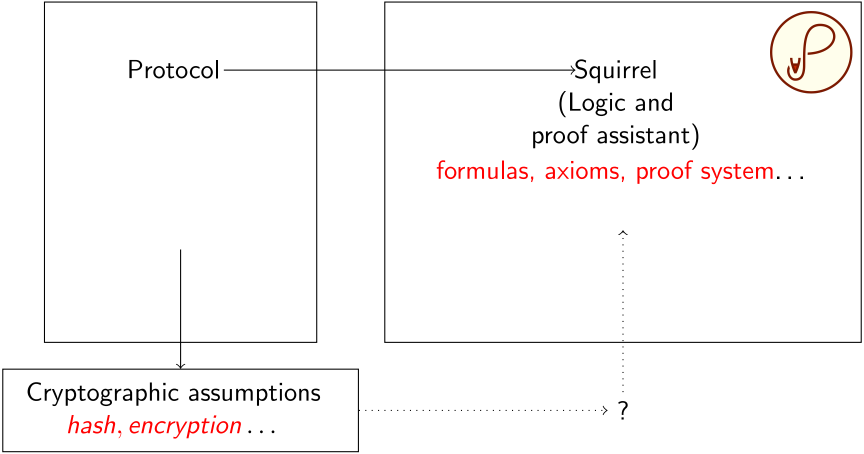
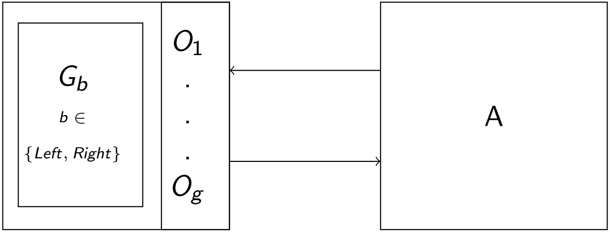
Context: Squirrel and Cryptographic assumptions



Cryptographic assumptions



Definition (Indinstinguishability)

For any polynomial-time and randomized algorithm A,

$$|\Pr(A^{G_{Left}} = 1) - \Pr(A^{G_{Right}} = 1)|$$

is negligible (*i.e.*, roughly exponentially small in the length of the keys).

Example: PRF games

Intuition: a pseudo random function is a function that "seems" random.

Example (PRF games)

Game G_{Left}

$Challenge(x)$:

return $\boxed{h(x,k)}$

Game G_{Righ}

$Challenge(x)$:

$r \xleftarrow{\$}$

return $\boxed{r}$

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Example (PRF games)

Game G_{Left} *Init* :

$k \xleftarrow{\$}$

Challenge(x) :

return $\boxed{h(x,k)}$

Game G_{Righ} *Init* :

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Challenge(x) :

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Example (PRF games)

Game G_{Left}	*Init* :	*Hash*(x) :	*Challenge*(x) :
	$k \xleftarrow{\$}$	
		return $h(x, k)$
			return $\boxed{h(x,k)}$

Game G_{Righ}	*Init* :	*Hash*(x) :	*Challenge*(x) :
	$k \xleftarrow{\$}$		$r \xleftarrow{\$}$
		return $h(x, k)$
			return $\boxed{r}$

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Example (PRF games)

Game G_{Left}	$Init$:	$Hash(x)$:	$Challenge(x)$:
	$k \xleftarrow{\$}$	$log := x :: log$	$r \xleftarrow{\$}$
	$log := []$	$return\ h(x, k)$	$if\ x \notin log$
			$\quad log := x :: log$
			$\quad return\ \boxed{h(x,k)}$

Game G_{Righ}	$Init$:	$Hash(x)$:	$Challenge(x)$:
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			$\quad log := x :: log$
			$\quad return\ \boxed{r}$

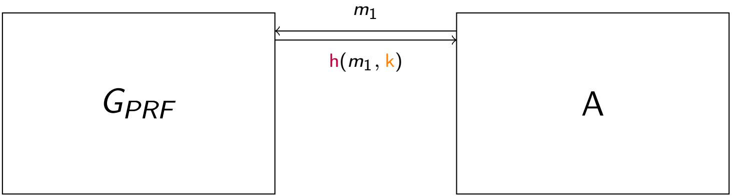
Example: PRF games

Example (PRF pair of games)

Game G_{PRF}

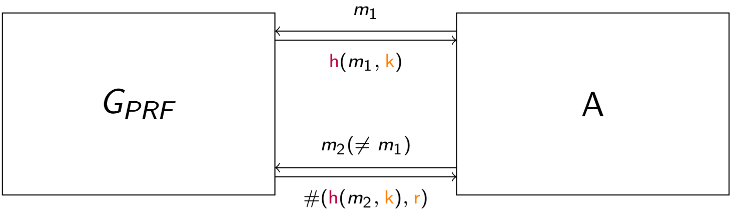
$Init$:	$Hash(x)$:	$Challenge(x)$:
$\mathsf{k} \xleftarrow{\$}$;	$L := x :: L$	$\mathsf{r} \xleftarrow{\$}$
$l := []$;	$\mathsf{h}(x, \mathsf{k})$	$if\ x \notin L$
		$L := x :: L$;
		$\boxed{\#(\mathsf{h}(\mathsf{x},\mathsf{k}),\mathsf{r})}$

Playing with PRF: sequence of messages



$$m_1, \mathsf{h}(m_1, \mathsf{k})$$

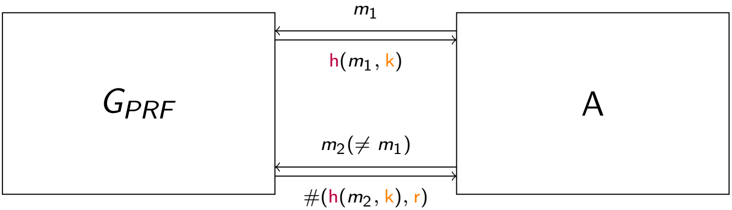
Playing with PRF: sequence of messages



$$m_1, h(m_1, k), m_2, \#(h(m_2, k), r)$$

$$:= (\ (m_1, h(m_1, k), m_2, h(m_2, k)),$$
$$(m_1, h(m_1, k), m_2, r)\ )$$

Playing with PRF: sequence of messages



$$\text{equiv}(m_1, h(m_1, k), m_2, \#(h(m_2, k), r))$$

If there exists an adversary that can distinguish between this two sequences of messages, then the PRF assumption doesn't hold.

Terms and formulas

Definition (Terms)

Intuition: terms represent messages

$$
\begin{aligned}
t := &\mid \mathsf{r} && \text{(names, repr. samplings)}\\
&\mid f(t_1, \ldots, t_n) && \text{(function application)}\\
&\mid \#(t_0, t_1) && \text{(left/right difference)}
\end{aligned}
$$

Definition (Equivalence formulas)

$$\mathsf{equiv}(\vec{t})$$

PRF axiom schema

Question: is this formula a consequence of PRF assumption?

$$\mathsf{equiv}((m_1, \mathsf{h}(m_1, \mathsf{k}), m_2, \#(\mathsf{h}(m_2, \mathsf{k}), \mathsf{r})))$$

- $m_1 = \mathsf{k}$: adversary must not directly access the key.
- $m_1 = m_2$: forbidden by the game.
- $m_1 = \mathsf{r}$: r must be fresh.

Definition (PRF axiom schema)

For all terms $\vec{t}$ verifying specific syntactic properties:

$$\frac{}{\mathsf{equiv}(\ \vec{t}, \#(h(m, \mathsf{k}), \mathsf{r})\ )}$$

Problems

Problems with this method

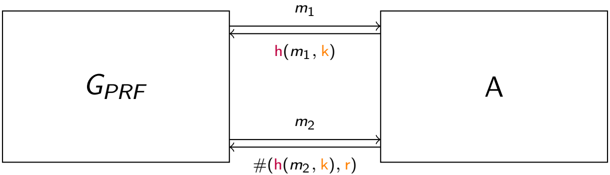
Ad hoc and manual work for each cryptographic axioms:

- Axiom schema design
- Correctness proof (understand the logic and its semantics)
- Implementation (understand the code)

Changing point of view

Input:

$$m_1, h(m_1, k), m_2, h(m_1, k)$$



Question: does there exists such an A?

Contributions

- Theoritical framework to reduce equivalences to cryptographic assumption: extended notion of bi-deduction [BDKM22].
- Proof system for bi-deduction
- Application: implementation of SQUIRREL tactic crypto

Bi-deduction

Construction of then bi-deduction judgement: simulator

$$\triangleright \vec{v}$$

means that there exists an adversary S such that $S^G() = \vec{v}$.

Link between Bi-deduction and Equivalence

If an adversary can compute $\vec{v}$ then the formula $\mathsf{equiv}(\vec{v})$ holds.

$$\textsc{Bi-deduction}\ \frac{\triangleright \vec{v}}{\mathsf{equiv}(\vec{v})}$$

What do we need ?

Goal: Framework for bi-deduction and associated proof system

Adversaries' capabilities ?

An adversary can:

- compute deterministic functions
- draw samplings
- interact with the game: oracles calls.

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- compute deterministic functions (done)
- draw samplings
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Definition

Function application: inference rule

$$\text{FA}\ \frac{\triangleright \vec{t} \quad adv(f)}{\triangleright f(\vec{t})}$$

S :

$\vec{x_t} := S_t()$

$y := f(\vec{x_t})$

return y

Samplings

Example

$$h(n, s), h(h(n, s), k)$$

simulator ← (n, s, n, s) | (k) → game

We need to keep track of the owner of each sampling.

Definition (Tags)

$$Tag = \{\mathtt{T}_S, \mathtt{T}^{\mathsf{glob}}_{\mathsf{G},key}, \ldots\}$$

$$n \leftarrow \mathtt{T}_S$$

$$s \leftarrow \mathtt{T}_S$$

$$k \leftarrow \mathtt{T}^{\mathsf{glob}}_{\mathsf{G},key}$$

Extending bi-deduction with constraints

Adding sampling tagging

C records who sampled what: $C : \triangleright \vec{v}$

Definition (Adversary samplings)

$$\frac{C : \triangleright \vec{v}}{C, \langle \mathsf{n}, \mathtt{T}_S \rangle : \triangleright \mathsf{n}, \vec{v}}\ \textsc{Adv sampling}$$

S :

$\vec{y} := S_v()$

$x := \$$

return $x, \vec{y}$

$$\dfrac{\dfrac{\dfrac{}{\emptyset : \triangleright \emptyset}}{\langle \mathsf{s}, \mathtt{T}_S \rangle : \triangleright \mathsf{s}}\ \textsc{Adv sampling}}{\dfrac{\langle \mathsf{n}, \mathtt{T}_S \rangle, \langle \mathsf{s}, \mathtt{T}_S \rangle : \triangleright \mathsf{n}, \mathsf{s}}{\langle \mathsf{n}, \mathtt{T}_S \rangle, \langle \mathsf{s}, \mathtt{T}_S \rangle : \triangleright \mathsf{h}(\mathsf{n}, \mathsf{s})}\ \mathrm{FA}}\ \textsc{Adv sampling}$$

Oracle calls on example

Definition (Oracle rule: instantiated for hash oracle)

$$\textsc{Hash}\ \frac{C : \rhd m, \vec{v}}{C, \langle \mathsf{k}, \mathrm{T}^{\mathsf{glob}}_{\mathsf{G},key} \rangle : \rhd \mathsf{h}(m, \mathsf{k}), \vec{v}}$$

S :

$x_m, \vec{x_v} := S_{m,v}()$

$x := \mathcal{O}_{Hash}(x_m)$

return $x, \vec{x_v}$

Example

$$\mathsf{h}(\mathsf{n}, \mathsf{s}), \mathsf{h}(\mathsf{h}(\mathsf{n}, \mathsf{s}), \mathsf{k})$$

$$\dfrac{\dfrac{\vdots}{C : \rhd \mathsf{h}(\mathsf{n}, \mathsf{s})}}{C, \langle \mathsf{k}, \mathrm{T}^{\mathsf{glob}}_{\mathsf{G},key} \rangle : \rhd \mathsf{h}(\mathsf{h}(\mathsf{n}, \mathsf{s}), \mathsf{k})}\ \textsc{Hash}$$

Oracle rule: Challenge

Oracle rule instantiated for challenge

$$\frac{C : \rhd m, \vec{v}}{C, \langle \mathsf{r}, \mathtt{T}^{\mathsf{loc}}_{\mathsf{G}} \rangle, \langle \mathsf{k}, \mathtt{T}^{\mathsf{glob}}_{\mathsf{G}, key} \rangle : \rhd \#(\mathsf{h}(m, \mathsf{k}), \mathsf{r}), \vec{v}}$$

$$m \longrightarrow \mathsf{h}(m, \mathsf{k}) \longrightarrow \#(\mathsf{h}(m, \mathsf{k}), \mathsf{r})$$

Oracle rule: Challenge

Oracle rule instantiated for challenge

$$\frac{\theta, \varphi, C : \rhd m, \vec{v} \qquad \{\varphi\}\mathcal{O}_{Challenge}(m)\{\psi\}}{\theta, \psi, C, \langle \mathsf{r}, \mathtt{T}^{\mathsf{loc}}_{\mathsf{G}} \rangle, \langle \mathsf{k}, \mathtt{T}^{\mathsf{glob}}_{\mathsf{G}, key} \rangle : \rhd \#(\mathsf{h}(m, \mathsf{k}), \mathsf{r}), \vec{v}}$$

$$m \longrightarrow \mathsf{h}(m, \mathsf{k}) \longrightarrow \#(\mathsf{h}(m, \mathsf{k}), \mathsf{r})$$

$$log = [m] \qquad log = [m, m]?$$

Adding pre and post conditions

$$\varphi, \psi; C : \rhd \vec{v}$$

Consistency of taggings

C: registers randomness usage.

We want to ensure:

- Not two samples for one "role" (*e.g.*, $k \leftarrow \mathtt{T}^{\mathsf{glob}}_{\mathsf{G},key}, k' \leftarrow \mathtt{T}^{\mathsf{glob}}_{\mathsf{G},key}$)
- No sample owned by both the adversary and the oracles
- Freshness of local random sampling (*e.g.*, r)

Definition (validity of C)

$\mathsf{Valid}(C) \stackrel{def}{=} \mathsf{Uniqueness}(C) \wedge \mathsf{Ownership}(C) \wedge \mathsf{Freshness}(C)$

Coming back to the judgment

Bideduction judgmenent

For all conditions φ, ψ, constraints C, term $\vec{v}$:

$$\varphi, \psi; C : \triangleright \vec{v} \text{ iff}$$

if Valid(C) there exists a G-adversary S such that
for all memory μ satisfying φ, $(\!| S |\!)_\mu() = \mu'$

- μ' satisfy ψ
- $\mu'[res] = \vec{v}$

Semantics

Denotational semantic for adversaries and terms with early-random samplings.

$$(\!|S|\!)^{\eta}_{\mu} = X^{\text{left}}_S, X^{\text{right}}_S$$
$$[\![\vec{v}]\!]^{\eta} = X^{\text{left}}_v, X^{\text{right}}_v$$

Different randomness sources:

$$X^{\text{left/right}}_S : p \mapsto \cdots$$
$$X^{\text{left/right}}_v : \rho \mapsto \cdots$$

Equality of distribution: $\Pr_p(X^b_S = x) = \Pr_\rho(X^b_s = x)$

Coupling

Lifting through couplings:

$$
\begin{array}{ccc}
X_S^{\text{left}} & \approx & X_S^{\text{right}} \\
| & & | \\
=_d & & =_d \\
| & & | \\
X_v^{\text{left}} & \sim & X_v^{\text{right}}
\end{array}
$$

(proof: coupling built from C)

Implementation : game declarations

A language for games :

```
game PRF = {
  rnd key : kty;
  var lhash : mset = empty_set; var lchal : mset = empty_set;
  oracle ohash (x:message) : message = {
    lhash := add x lhash; return if mem x lchal then zero else h (x, key)
  }
  oracle challenge (x:message) : message = {
    rnd r : message;
    var old_lchal : mset = lchal;
    lchal := add x lchal;
    return if (mem x old_lchal || mem x lhash) then zero
           else diff(r, h (x, key))
  }
}
```

Implementation: tactic crypto

- Goal-directed proof-search procedure, based on the proof system.
- Langage to describe game memory.
- Ad hoc handling of induction for recursive terms.

Case studies

Protocol	Hypothesis	Properties
Basic Hash	EUF-MAC and PRF	Unlinkability
Hash Lock	PRF	Strong secrecy
Private Authentification	$\mathrm{CCA}_{\$}$	Anonymity
NSL proof step	CCA2	Strong secrecy

Conclusion

Contributions:

- Formal framework linking games, adversaries, and formulas
- Bi-deduction judgment to capture adversaries interacting with a game
- Proof system for this judgment
- Implementation: proof search automation and SQUIRREL tactics
- Validation through various case studies

Future work:

- Larger case study: FOO e-voting protocol
- Limitation in the theory.
- Improve tactic heuristics (time insentive invariant).

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(Submitted work to CSF'24.)

References

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